Lecture 2: August 28

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Finite-dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ play a central role in Hodge theory. We should therefore spend some time reviewing their basic properties. Let me remind you that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is 3-dimensional, with basis given by the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They are subject to the following three relations:

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

We already saw last time that every finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a direct sum of irreducible representations. Let us now analyze the structure of irreducible representations. Suppose then that V is a finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Choose a nonzero eigenvector $v \in V$ for H, say with $Hv = \lambda v$. For the time being, $\lambda \in \mathbb{C}$ can be any complex number, but we will see in a moment that λ actually has to be an integer. From the two relations $[H, X] = 2X$ and $[H, Y] = -2Y$, we get

$$\begin{aligned} H(Xv) &= [H, X]v + X(Hv) = 2Xv + X(\lambda v) = (\lambda + 2)Xv, \\ H(Yv) &= [H, Y]v + Y(Hv) = -2Yv + Y(\lambda v) = (\lambda - 2)Yv, \end{aligned}$$

and so X maps the λ-eigenspace $E_\lambda(H)$ into $E_{\lambda+2}(H)$, whereas Y maps $E_\lambda(H)$ into $E_{\lambda-2}(H)$. In particular, the vectors $v, Yv, Y^2v, \dots$ are linearly independent, and since V is finite-dimensional, this means that $Y^n v = 0$ for large n. After replacing v by its image under a suitable power of Y, we may therefore assume in addition that $Yv = 0$. So from now on, $Hv = \lambda v$ and $Yv = 0$.

Now consider the sequence of vectors

$$v, Xv, X^2v, \dots, X^n v, \dots$$

The action of X takes each vector to the next one; the following lemma describes the action of Y.

Lemma 2.1. *If $Hv = \lambda v$ and $Yv = 0$, then one has*

$$\begin{aligned} YX^n v &= -n(\lambda + n - 1)X^{n-1}v, \\ Y^n X^n v &= (-1)^n n! \cdot \lambda(\lambda + 1) \cdots (\lambda + n - 1)v, \end{aligned}$$

for every $n \geq 0$.

Proof. We prove the first identity by induction on $n \geq 0$, with the case $n = 0$ being trivial. For $n \geq 0$, the relation $[X, Y] = H$ gives

$$\begin{aligned} YX^{n+1}v &= -HX^n v + XYX^n v = -(\lambda + 2n)X^n v - n(\lambda + n - 1)X^n v \\ &= -(n + 1)(\lambda + n)X^n v, \end{aligned}$$

as required. This gives

$$Y^{n+1}X^{n+1}v = -(n + 1)(\lambda + n)Y^n X^n v,$$

and so the second identity again follows by induction on $n \geq 0$. □

Now let $n \geq 0$ be the least integer such that $X^n v \neq 0$ and $X^{n+1}v = 0$; this exists for the same reason as above. The lemma shows that the subspace generated by $v, Xv, \dots, X^n v$ is stable under the action of H, X, and Y; since V is an irreducible representation, it follows that

$$V = \mathbb{C}\langle v, Xv, \dots, X^n v \rangle,$$

and therefore $\dim V = n+1$. Now let us show that $\lambda = -n$. Again from the lemma,

$$0 = Y^{n+1}X^{n+1}v = (-1)^{n+1}(n+1)! \cdot \lambda(\lambda+1)\cdots(\lambda+n-1)(\lambda+n)v,$$

and so λ has to be one of the integers $0, -1, \dots, -n$. Now if $\lambda \neq -n$, then

$$Y^n X^n v = (-1)^n n! \cdot \lambda(\lambda+1)\cdots(\lambda+n-1)v = 0,$$

and so the subspace generated by $Xv, X^2v, \dots, X^n v$ would also be stable under the action of H, X, and Y; but since V is irreducible, this cannot be. Therefore $\lambda = -n$ is the only possibility.

To summarize: if V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with $\dim V = n+1$, then there exists a nonzero vector $v \in V$ with $Hv = -nv$ and $Yv = 0$, and

$$V = \mathbb{C}\langle v, Xv, \dots, X^n v\rangle.$$

In this basis, the action of H is diagonal, with eigenvalues $-n, -n+2, \dots, n$; the action of X is the obvious one; and the action of Y is described by the lemma. In particular, $\mathfrak{sl}_2(\mathbb{C})$ has, up to isomorphism, a unique irreducible representation of every finite dimension.

Since every finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a direct sum of irreducible representations, we can draw the following conclusions:

(1) If V is a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$, then the action of H is diagonalizable, and all eigenvalues of H are integers. If we set $V_k = E_k(H)$, for $k \in \mathbb{Z}$, this means that

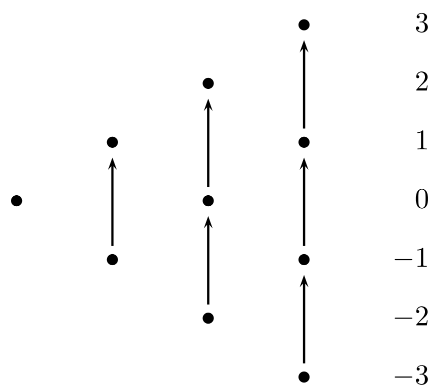
$$V = \bigoplus_{k\in\mathbb{Z}} V_k,$$

and so V becomes a graded vector space. Moreover, $XV_k \subseteq V_{k+2}$ and $YV_k \subseteq V_{k-2}$.

(2) For every $k \geq 1$, the operator X^k gives an isomorphism between V_{-k} and V_k. This is clearly true for irreducible representations, and therefore in general. In particular, $\dim V_{-k} = \dim V_k$, and so the entire representation is symmetric around the piece of weight 0.

(3) Every irreducible representation is generated by a "primitive vector", meaning a vector in the kernel of Y; these are also sometimes called "vectors of lowest weight".

The following schematic picture shows a typical representation of $\mathfrak{sl}_2(\mathbb{C})$:



Each dot stands for a one-dimensional subspace, the vertical arrows indicate the action of X, and the numbers on the right are the eigenvalues of H, which are usually called the *weights* of the representation. Note again the symmetry around the center.

We have just seen that an irreducible representation has a basis consisting of a primitive vector and its images under powers of X. This fact generalizes to arbitrary finite-dimensional representations as follows.

Proposition 2.2. *Let V be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$, and set $V_k = E_k(H)$. Every vector $v \in V_k$ has a unique decomposition*

$$v = \sum_{j \geq \max(k,0)} \frac{X^j}{j!} v_j$$

with $v_j \in V_{k-2j} \cap \ker Y$, called the Lefschetz decomposition.

Proof. We did not prove this in class, but I will write down a proof for the sake of completeness. Let me first explain where the range of the summation comes from. We have seen above that if $v \in V$ is a nonzero vector such that $Hv = -nv$ and $Yv = 0$, then one has $X^n v \neq 0$ and $X^{n+1} v = 0$. In the case of $v_j \in V_{k-2j}$, this says that $X^{2j-k+1} v = 0$, and so $X^j v_j = 0$ whenever $j \geq 2j - k + 1$. This is why only terms with $j \geq k$ appear.

The existence of a Lefschetz decomposition is easy: every finite-dimensional representation is a direct sum of irreducible ones, and for an irreducible representation, the result is obvious. See also the picture above.

To prove the uniqueness of the decomposition, it is enough to show that if

$$0 = \sum_{j \geq 0} \frac{X^j}{j!} v_j \in V_k$$

for certain vectors $v_j \in V_{k-2j} \cap \ker Y$, then actually $X^j v_j = 0$ for every j. Since $V_k = 0$ for $k \gg 0$, we can argue by descending induction on k. After applying X to the sum, we get

$$0 = \sum_{j \geq 0} \frac{X^{j+1}}{j!} v_j \in V_{k+2},$$

and therefore $X^{j+1} v_j = 0$ by induction. But $v_j \in V_{k-2j}$ is a primitive vector, and so $X^{2j-k} v_j = 0$ implies that $v_j = 0$. The conclusion is that $v_j = 0$ for $2j - k \geq j+1$, hence for $j \geq k+1$. Since we have already seen that $X^j v_j = 0$ for $j \leq k-1$, this leaves at most the term $X^k v_k / k!$ (if $k \geq 0$), which must therefore also be zero. ∎

The Lefschetz decomposition reduces most questions about arbitrary vectors in an $\mathfrak{sl}_2(\mathbb{C})$-representation to the special case of primitive vectors.

We have seen above that finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ is a direct sum of weight spaces, which are symmetric around weight 0. There is another useful way to see this symmetry, involving the Lie group $\mathrm{SL}_2(\mathbb{C})$. A basic fact is that every finite-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ lifts to a representation of the Lie group $\mathrm{SL}_2(\mathbb{C})$. The lifting is compatible with the exponential map, in the following way. If $M \in \mathfrak{sl}_2(\mathbb{C})$, then

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} \in \mathrm{SL}_2(\mathbb{C}),$$

and the element e^M acts on the representation as $\mathrm{id} + M + \frac{1}{2} M^2 + \cdots$. Now consider the so-called *Weyl element*

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

A brief computation shows that $w = e^X e^{-Y} e^X$, and so one can use the exponential series to see how w acts on representations of $\mathfrak{sl}_2(\mathbb{C})$.

The point of introducing w is that it explains the symmetry between the weight spaces V_k and V_{-k}. Namely, the Lie group $\mathrm{SL}_2(\mathbb{C})$ acts on its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by conjugation, and under this action, one has

$$wXw^{-1} = -Y, \quad wYw^{-1} = -X, \quad wHw^{-1} = -H.$$

This can be checked by direct computation: for example,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The identity $wHw^{-1} = -H$ means that the action of w interchanges the two weight spaces V_k and V_{-k}, which must therefore be of the same dimension.

Note. The element

$$w^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

acts on the weight space V_k as $(-1)^k$, and *not* as multiplication by -1. Symbolically, one has the identity $w^2 = (-1)^H$, where the right-hand side is a convenient abbreviation for the power series $e^{i\pi H}$.

If $v \in V_{-n}$ is a primitive vector, we can compute $wv \in V_n$ with the help of the formula $w = e^X e^{-Y} e^X$. Indeed,

$$we^{-X} v = e^X e^{-Y} v = e^X v,$$

using $Yv = 0$ in the second step. Expanding both sides into power series gives

$$w \sum_{j=0}^{\infty} (-1)^j \frac{X^j}{j!} v = \sum_{j=0}^{\infty} \frac{X^j}{j!} v,$$

and after projecting to the weight space V_n, we find that

$$wv = \frac{X^n}{n!} v. \tag{2.3}$$

This shows that, at least as far as the action of w is concerned, the most natural basis for the irreducible representation generated by v is

$$v, Xv, \frac{X^2}{2!} v, \dots, \frac{X^n}{n!} v.$$

If we instead project to the weight space V_{n-2j}, we get the useful identity

$$w \frac{X^j}{j!} v = (-1)^j \frac{X^{n-j}}{(n-j)!} v, \tag{2.4}$$

valid for any $v \in V_{-n} \cap \ker Y$.

Exercise 2.1. Show that the identity in (2.4) is actually symmetric in j and $n - j$.

We can use the Lefschetz decomposition to get a simple formula for the action of w on arbitrary vectors.

Proposition 2.5. *Write the Lefschetz decomposition of $v \in V_k$ as*

$$v = \sum_{j \geq \max(k,0)} \frac{X^j}{j!} v_j$$

with $v_j \in V_{k-2j} \cap \ker Y$ primitive. Then one has

$$wv = \sum_{j \geq \max(k,0)} (-1)^j \frac{X^{j-k}}{(j-k)!} v_j.$$

Proof. Since $v_j \in V_{k-2j}$ is primitive, this follows from (2.4) by taking $n = 2j - k$. $\square$

Weil's identity and polarizations. We return to the example of compact Kähler manifolds. Let X be a compact Kähler manifold of dimension n, with Kähler form ω. Before we get to constructing a polarization on the cohomology of X, I first want to make some comments about the choice of $i = \sqrt{-1}$. (This is a topic that my graduate students are already familiar with.) You may remember that

$$\int_X \frac{\omega^n}{n!} = \mathrm{vol}(X)$$

is the volume of X with respect to the chosen Kähler metric h. The metric itself has nothing to do with the square root of -1, but the Kähler form is obtained by taking the imaginary part of the metric, viewed as a 2-tensor, and the imaginary part of a complex number depends on i. We can fix this problem by working with

$$2\pi i\, \omega$$

which does not depend on the choice of $i = \sqrt{-1}$. (The factor 2π is there to make the formulas in the geometric case nicer.)

Example 2.6. On $\mathbb{P}^1$ with the Fubini-Study metric, $2\pi i\,\omega$ is equal to the first Chern class of the line bundle $\mathscr{O}_{\mathbb{P}^1}(1)$.

Although not apparent from the notation, the integral also depends on the choice of i. The reason is that integration requires an orientation of the underlying smooth manifold of X, and the standard orientation of $\mathbb{C}$ is the one in which $1, i$ forms a positively-oriented basis. We can fix this problem by working with

$$\frac{1}{(2\pi i)^n} \int_X$$

which is independent of the choice of i. (Replacing i by $-i$ changes the orientation of $\mathbb{C}$ by a factor of -1, hence the orientation of the n-dimensional complex manifold X by $(-1)^n$, but this change is absorbed by denominator.) So a better way to write the above formula for the volume is

$$\frac{1}{(2\pi i)^n} \int_X \frac{(2\pi i\,\omega)^n}{n!} = \mathrm{vol}(X),$$

because now everything is independent of i.

Having said this, we should redefine the $\mathfrak{sl}_2(\mathbb{C})$-action on the cohomology of X. Namely, we should let $X \in \mathfrak{sl}_2(\mathbb{C})$ act as

$$X = 2\pi i\, L \colon A^k(X, \mathbb{C}) \to A^{k+2}(X, \mathbb{C})$$

and we should let $Y \in \mathfrak{sl}_2(\mathbb{C})$ act as

$$Y = \frac{1}{2\pi i} \Lambda \colon A^k(X, \mathbb{C}) \to A^{k-2}(X, \mathbb{C}).$$

Their commutator $H = [X, Y]$ still acts as multiplication by $k - n$ on $A^k(X, \mathbb{C})$, and so we obtain an infinite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ on the space of forms, and a finite-dimensional representation on the cohomology of X. Even though the space of all forms is infinite-dimensional, the subrepresentation generated by any particular form is of course finite-dimensional, and so the Lefschetz decomposition also exists on the level of forms.

Consider again the Weyl element $w = e^X e^{-Y} e^X \in \mathrm{SL}_2(\mathbb{C})$. It maps $H^{n-k}(X, \mathbb{C})$ isomorphically to $H^{n+k}(X, \mathbb{C})$, exactly like the Hodge $*$-operator. This suggests that the two operators should be related.

Proposition 2.7. *For every $\alpha \in A^{p,q}(X)$, one has*

$$*\alpha = (2\pi)^k \cdot \frac{(-1)^q \varepsilon(k)}{(2\pi i)^n} \cdot w\alpha,$$

where $k = p + q$ *and* $\varepsilon(k) = (-1)^{k(k-1)/2}$.

Proof. To simplify the notation, set $k = p + q$. Suppose first that $Y\alpha = 0$. Then

$$w\alpha = \frac{X^{n-k}}{(n-k)!}\alpha$$

by virtue of (2.3). If we substitute this into Weil's identity (1.8), we get

$$*\alpha = i^{q-p}\varepsilon(k)\frac{L^{n-k}}{(n-k)!}\alpha = \frac{i^{q-p}\varepsilon(k)}{(2\pi i)^{n-k}}\frac{X^{n-k}}{(n-k)!}\alpha = (2\pi)^k \cdot \frac{(-1)^q\varepsilon(k)}{(2\pi i)^n} \cdot w\alpha,$$

as claimed. The general case follows from this special case by using the Lefschetz decomposition. Namely, we have $*L = \Lambda *$, and therefore

$$* X = (2\pi i)^2 Y * = -(2\pi)^2 Y * .$$

At the same time, $wXw^{-1} = -Y$, and therefore

$$wX = -Yw.$$

Now write the Lefschetz decomposition of $\alpha \in A^{p,q}(X)$ as

$$\alpha = \sum_{j \geq \max(k,0)} \frac{X^j}{j!}\alpha_j,$$

where $\alpha_j \in A^{p-j,q-j}(X) \cap \ker Y$ is primitive. Then

$$\begin{aligned} * \alpha &= \sum_{j \geq \max(k,0)} (-1)^j (2\pi)^{2j} \frac{Y^j}{j!} *\alpha_j \\ &= \sum_{j \geq \max(k,0)} (-1)^j (2\pi)^{2j} \frac{Y^j}{j!} (2\pi)^{k-2j} \frac{(-1)^{q-j}\varepsilon(k-2j)}{(2\pi i)^n} w\alpha_j. \end{aligned}$$

Using the fact that $\varepsilon(k-2j) = (-1)^j\varepsilon(k)$, we can rewrite this as

$$\begin{aligned} * \alpha &= (2\pi)^k \frac{(-1)^q\varepsilon(k)}{(2\pi i)^n} \sum_{j \geq \max(k,0)} (-1)^j \frac{Y^j}{j!} w\alpha_j \\ &= (2\pi)^k \frac{(-1)^q\varepsilon(k)}{(2\pi i)^n} \sum_{j \geq \max(k,0)} w\frac{X^j}{j!}\alpha_j = (2\pi)^k \frac{(-1)^q\varepsilon(k)}{(2\pi i)^n} \cdot w\alpha. \end{aligned}$$

This is the desired formula. $\square$